

# **JAM 2006**

**MATHEMATICAL STATISTICS TEST PAPER**

## Special Instructions / Useful Data

1. For an event  $A$ ,  $P(A)$  denotes the probability of the event  $A$ .
2. The complement of an event is denoted by putting a superscript “ $c$ ” on the event, e.g.  $A^c$  denotes the complement of the event  $A$ .
3. For a random variable  $X$ ,  $E(X)$  denotes the expectation of  $X$  and  $V(X)$  denotes its variance.
4.  $N(\mu, \sigma^2)$  denotes a normal distribution with mean  $\mu$  and variance  $\sigma^2$ .
5. Standard normal random variable is a random variable having a normal distribution with mean 0 and variance 1.
6.  $P(Z > 1.96) = 0.025$ ,  $P(Z > 1.65) = 0.050$ ,  $P(Z > 0.675) = 0.250$  and  $P(Z > 2.33) = 0.010$ , where  $Z$  is a standard normal random variable.
7.  $P(\chi^2 \geq 9.21) = 0.01$ ,  $P(\chi^2 \geq 0.02) = 0.99$ ,  $P(\chi^2 \geq 11.34) = 0.01$ ,  $P(\chi^2 \geq 9.49) = 0.05$ ,  
 $P(\chi^2 \geq 0.71) = 0.95$ ,  $P(\chi^2 \geq 11.07) = 0.05$  and  $P(\chi^2 \geq 1.15) = 0.95$ , where  $P(\chi_n^2 \geq c) = \alpha$ , where  $\chi_n^2$  has a Chi-square distribution with  $n$  degrees of freedom.
8.  $n!$  denotes the factorial of  $n$ .
9. The determinant of a square matrix  $A$  is denoted by  $|A|$ .
10.  $\mathbb{R}$ : The set of all real numbers.
11.  $\mathbb{R}^n$ :  $n$ -dimensional Euclidean space.
12.  $y'$  and  $y''$  denote the first and second derivatives respectively of the function  $y(x)$  with respect to  $x$ .

**NOTE: This Question-cum-Answer book contains THREE sections, the Compulsory Section A, and the Optional Sections B and C.**

- Attempt ALL questions in the compulsory section A. It has 15 objective type questions of six marks each and also *nine* subjective type questions of *fifteen* marks each.
- Optional Sections B and C have *five* subjective type questions of *fifteen* marks each.
- Candidates seeking admission to either of the two programmes, M.Sc. in Applied Statistics & Informatics at IIT Bombay and M.Sc. in Statistics & Informatics at IIT Kharagpur, are required to attempt ONLY Section B (Mathematics) from the Optional Sections.
- Candidates seeking admission to the programme, M.Sc. in Statistics at IIT Kanpur, are required to attempt ONLY Section C (Statistics) from the Optional Sections.

You must therefore attempt either Optional Section B or Optional Section C depending upon the programme(s) you are seeking admission to, and accordingly tick one of the boxes given below.

Optional Section Attempted	B	
	C	

- *The negative marks for the Objective type questions will be carried over to the total marks.*
- Write the answers to the objective questions in the Answer Table for Objective Questions provided on page MS 11/63 only.

## Compulsory Section A

1. If  $a_n > 0$  for  $n \geq 1$  and  $\lim_{n \rightarrow \infty} (a_n)^{1/n} = L < 1$ , then which of the following series is not convergent?
- (A)  $\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}}$
- (B)  $\sum_{n=1}^{\infty} a_n^2$
- (C)  $\sum_{n=1}^{\infty} \sqrt{a_n}$
- (D)  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{a_n}}$
2. Let  $E$  and  $F$  be two mutually disjoint events. Further, let  $E$  and  $F$  be independent of  $G$ . If  $p = P(E) + P(F)$  and  $q = P(G)$ , then  $P(E \cup F \cup G)$  is
- (A)  $1 - pq$
- (B)  $q + p^2$
- (C)  $p + q^2$
- (D)  $p + q - pq$
3. Let  $X$  be a continuous random variable with the probability density function symmetric about 0. If  $V(X) < \infty$ , then which of the following statements is true?
- (A)  $E(|X|) = E(X)$
- (B)  $V(|X|) = V(X)$
- (C)  $V(|X|) < V(X)$
- (D)  $V(|X|) > V(X)$
4. Let
- $$f(x) = x|x| + |x-1|, \quad -\infty < x < \infty.$$
- Which of the following statements is true?
- (A)  $f$  is not differentiable at  $x=0$  and  $x=1$ .
- (B)  $f$  is differentiable at  $x=0$  but not differentiable at  $x=1$ .
- (C)  $f$  is not differentiable at  $x=0$  but differentiable at  $x=1$ .
- (D)  $f$  is differentiable at  $x=0$  and  $x=1$ .
5. Let  $A\tilde{x} = \tilde{b}$  be a non-homogeneous system of linear equations. The augmented matrix  $[A : \tilde{b}]$  is given by

$$\left[ \begin{array}{cccc|c} 1 & 1 & -2 & 1 & 1 \\ -1 & 2 & 3 & -1 & 0 \\ 0 & 3 & 1 & 0 & -1 \end{array} \right].$$

Which of the following statements is true?

- (A) Rank of  $A$  is 3.
- (B) The system has no solution.
- (C) The system has unique solution.
- (D) The system has infinite number of solutions.

6. An archer makes 10 independent attempts at a target and his probability of hitting the target at each attempt is  $\frac{5}{6}$ . Then the conditional probability that his last two attempts are successful given that he has a total of 7 successful attempts is

- (A)  $\frac{1}{5^5}$
- (B)  $\frac{7}{15}$
- (C)  $\frac{25}{36}$
- (D)  $\frac{8!}{3! 5!} \left(\frac{5}{6}\right)^7 \left(\frac{1}{6}\right)^3$

7. Let

$$f(x) = (x-1)(x-2)(x-3)(x-4)(x-5), \quad -\infty < x < \infty.$$

The number of distinct real roots of the equation  $\frac{d}{dx} f(x) = 0$  is exactly

- (A) 2
- (B) 3
- (C) 4
- (D) 5

8. Let

$$f(x) = \frac{k|x|}{(1+|x|)^4}, \quad -\infty < x < \infty.$$

Then the value of  $k$  for which  $f(x)$  is a probability density function is

- (A)  $\frac{1}{6}$
- (B)  $\frac{1}{2}$
- (C) 3
- (D) 6

9. If  $M_X(t) = e^{3t+8t^2}$  is the moment generating function of a random variable  $X$ , then  $P(-4.84 < X \leq 9.60)$  is

- (A) equal to 0.700
- (B) equal to 0.925
- (C) equal to 0.975
- (D) greater than 0.999

10. Let  $X$  be a binomial random variable with parameters  $n$  and  $p$ , where  $n$  is a positive integer and  $0 \leq p \leq 1$ . If  $\alpha = P(|X - np| \geq \sqrt{n})$ , then which of the following statements holds true for all  $n$  and  $p$ ?

(A)  $0 \leq \alpha \leq \frac{1}{4}$

(B)  $\frac{1}{4} < \alpha \leq \frac{1}{2}$

(C)  $\frac{1}{2} < \alpha < \frac{3}{4}$

(D)  $\frac{3}{4} \leq \alpha \leq 1$

11. Let  $X_1, X_2, \dots, X_n$  be a random sample from a Bernoulli distribution with parameter  $p$ ;  $0 \leq p \leq 1$ . The

bias of the estimator  $\frac{\sqrt{n} + 2 \sum_{i=1}^n X_i}{2(n + \sqrt{n})}$  for estimating  $p$  is equal to

(A)  $\frac{1}{\sqrt{n} + 1} \left( p - \frac{1}{2} \right)$

(B)  $\frac{1}{n + \sqrt{n}} \left( \frac{1}{2} - p \right)$

(C)  $\frac{1}{\sqrt{n} + 1} \left( \frac{1}{2} + \frac{p}{\sqrt{n}} \right) - p$

(D)  $\frac{1}{\sqrt{n} + 1} \left( \frac{1}{2} - p \right)$

12. Let the joint probability density function of  $X$  and  $Y$  be

$$f(x, y) = \begin{cases} e^{-x}, & \text{if } 0 \leq y \leq x < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

Then  $E(X)$  is

(A) 0.5

(B) 1

(C) 2

(D) 6

13. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined as

$$f(t) = \begin{cases} \frac{\tan t}{t}, & t \neq 0, \\ 1, & t = 0. \end{cases}$$

Then the value of  $\lim_{x \rightarrow 0} \frac{1}{x^2} \int_{x^2}^{x^3} f(t) dt$

- (A) is equal to  $-1$
- (B) is equal to  $0$
- (C) is equal to  $1$
- (D) does not exist

14. Let  $X$  and  $Y$  have the joint probability mass function;

$$P(X = x, Y = y) = \frac{1}{2^{y+2}(y+1)} \left( \frac{2y+1}{2y+2} \right)^x, \quad x, y = 0, 1, 2, \dots$$

Then the marginal distribution of  $Y$  is

- (A) Poisson with parameter  $\lambda = \frac{1}{4}$
- (B) Poisson with parameter  $\lambda = \frac{1}{2}$
- (C) Geometric with parameter  $p = \frac{1}{4}$
- (D) Geometric with parameter  $p = \frac{1}{2}$

15. Let  $X_1, X_2$  and  $X_3$  be a random sample from a  $N(3, 12)$  distribution. If  $\bar{X} = \frac{1}{3} \sum_{i=1}^3 X_i$  and

$S^2 = \frac{1}{2} \sum_{i=1}^3 (X_i - \bar{X})^2$  denote the sample mean and the sample variance respectively, then

$P(1.65 < \bar{X} \leq 4.35, 0.12 < S^2 \leq 55.26)$  is

- (A) 0.49
- (B) 0.50
- (C) 0.98
- (D) none of the above

16. (a) Let  $X_1, X_2, \dots, X_n$  be a random sample from an exponential distribution with the probability density function;

$$f(x; \theta) = \begin{cases} \theta e^{-\theta x}, & \text{if } x > 0, \\ 0, & \text{otherwise,} \end{cases}$$

where  $\theta > 0$ . Obtain the maximum likelihood estimator of  $P(X > 10)$ . **9 Marks**

(b) Let  $X_1, X_2, \dots, X_n$  be a random sample from a discrete distribution with the probability mass function given by

$$P(X = 0) = \frac{1-\theta}{2}; \quad P(X = 1) = \frac{1}{2}; \quad P(X = 2) = \frac{\theta}{2}, \quad 0 \leq \theta \leq 1.$$

Find the method of moments estimator for  $\theta$ . **6 Marks**

17. (a) Let  $A$  be a non-singular matrix of order  $n$  ( $n > 1$ ), with  $|A| = k$ . If  $adj(A)$  denotes the adjoint of the matrix  $A$ , find the value of  $|adj(A)|$ . **6 Marks**

(b) Determine the values of  $a, b$  and  $c$  so that  $(1, 0, -1)$  and  $(0, 1, -1)$  are eigenvectors of the matrix,

$$\begin{bmatrix} 2 & 1 & 1 \\ a & 3 & 2 \\ 3 & b & c \end{bmatrix}.$$

**9 Marks**

18. (a) Using Lagrange's mean value theorem, prove that

$$\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2},$$

where  $0 < \tan^{-1} a < \tan^{-1} b < \frac{\pi}{2}$ .

**6 Marks**

(b) Find the area of the region in the first quadrant that is bounded by  $y = \sqrt{x}$ ,  $y = x - 2$  and the  $x$ -axis.

**9 Marks**

19. Let  $X$  and  $Y$  have the joint probability density function;

$$f(x, y) = \begin{cases} c x y e^{-(x^2+2y^2)}, & \text{if } x > 0, y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Evaluate the constant  $c$  and  $P(X^2 > Y^2)$ .

20. Let  $PQ$  be a line segment of length  $\beta$  and midpoint  $R$ . A point  $S$  is chosen at random on  $PQ$ . Let  $X$ , the distance from  $S$  to  $P$ , be a random variable having the uniform distribution on the interval  $(0, \beta)$ . Find the probability that  $PS, QS$  and  $PR$  form the sides of a triangle.

21. Let  $X_1, X_2, \dots, X_n$  be a random sample from a  $N(\mu, 1)$  distribution. For testing  $H_0: \mu = 10$  against

$H_1: \mu = 11$ , the most powerful critical region is  $\bar{X} \geq k$ , where  $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$ . Find  $k$  in terms of  $n$  such

that the size of this test is 0.05.

Further determine the minimum sample size  $n$  so that the power of this test is at least 0.95.

22. Consider the sequence  $\{s_n\}$ ,  $n \geq 1$ , of positive real numbers satisfying the recurrence relation

$$s_{n-1} + s_n = 2 s_{n+1} \text{ for all } n \geq 2.$$

(a) Show that  $|s_{n+1} - s_n| = \frac{1}{2^{n-1}} |s_2 - s_1|$  for all  $n \geq 1$ .

(b) Prove that  $\{s_n\}$  is a convergent sequence.

23. The cumulative distribution function of a random variable  $X$  is given by

$$F(x) = \begin{cases} 0, & \text{if } x < 0, \\ \frac{1}{5} (1 + x^3), & \text{if } 0 \leq x < 1, \\ \frac{1}{5} [3 + (x-1)^2], & \text{if } 1 \leq x < 2, \\ 1, & \text{if } x \geq 2. \end{cases}$$

Find  $P(0 < X < 2)$ ,  $P(0 \leq X \leq 1)$  and  $P\left(\frac{1}{2} \leq X \leq \frac{3}{2}\right)$ .

24. Let  $A$  and  $B$  be two events with  $P(A|B) = 0.3$  and  $P(A|B^c) = 0.4$ . Find  $P(B|A)$  and  $P(B^c|A^c)$  in terms of  $P(B)$ . If  $\frac{1}{4} \leq P(B|A) \leq \frac{1}{3}$  and  $\frac{1}{4} \leq P(B^c|A^c) \leq \frac{9}{16}$ , then determine the value of  $P(B)$ .

### Optional Section B

25. Solve the initial value problem

$$y' - y + y^2 (x^2 + 2x + 1) = 0, \quad y(0) = 1.$$

26. Let  $y_1(x)$  and  $y_2(x)$  be the linearly independent solutions of

$$x y'' + 2 y' + x e^x y = 0.$$

If  $W(x) = y_1(x) y_2'(x) - y_2(x) y_1'(x)$  with  $W(1) = 2$ , find  $W(5)$ .

27. (a) Evaluate  $\int_0^1 \int_y^1 x^2 e^{xy} dx dy$ .

**9 Marks**

(b) Evaluate  $\iiint_W z dx dy dz$ , where  $W$  is the region bounded by the planes  $x = 0$ ,  $y = 0$ ,  $z = 0$ ,  $z = 1$  and the cylinder  $x^2 + y^2 = 1$  with  $x \geq 0$ ,  $y \geq 0$ .

**6 Marks**

28. A linear transformation  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$  is given by

$$T(x, y, z) = (3x + 11y + 5z, x + 8y + 3z).$$

Determine the matrix representation of this transformation relative to the ordered bases  $\{(1, 0, 1), (0, 1, 1), (1, 0, 0)\}$ ,  $\{(1, 1), (1, 0)\}$ . Also find the dimension of the null space of this transformation.

29. (a) Let  $f(x, y) = \begin{cases} \frac{x^2 + y^2}{x + y}, & \text{if } x + y \neq 0, \\ 0, & \text{if } x + y = 0. \end{cases}$

Determine if  $f$  is continuous at the point  $(0, 0)$ .

**6 Marks**

(b) Find the minimum distance from the point  $(1, 2, 0)$  to the cone  $z^2 = x^2 + y^2$ .

**9 Marks**

### Optional Section C

30. Let  $X_1, X_2, \dots, X_n$  be a random sample from an exponential distribution with the probability density function;

$$f(x; \theta) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}}, & \text{if } x > 0, \\ 0, & \text{otherwise,} \end{cases}$$

where  $\theta > 0$ . Derive the Cramér-Rao lower bound for the variance of any unbiased estimator of  $\theta$ .

Hence, prove that  $T = \frac{1}{n} \sum_{i=1}^n X_i$  is the uniformly minimum variance unbiased estimator of  $\theta$ .

31. Let  $X_1, X_2, \dots$  be a sequence of independently and identically distributed random variables with the probability density function;

$$f(x) = \begin{cases} \frac{1}{2} x^2 e^{-x}, & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Show that  $\lim_{n \rightarrow \infty} P(X_1 + \dots + X_n \geq 3(n - \sqrt{n})) \geq \frac{1}{2}$ .

32. Let  $X_1, X_2, \dots, X_n$  be a random sample from a  $N(\mu, \sigma^2)$  distribution, where both  $\mu$  and  $\sigma^2$  are unknown. Find the value of  $b$  that minimizes the mean squared error of the estimator

$$T_b = \frac{b}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2 \text{ for estimating } \sigma^2, \text{ where } \bar{X} = \frac{1}{n} \sum_{i=1}^n X_i.$$

33. Let  $X_1, X_2, \dots, X_5$  be a random sample from a  $N(2, \sigma^2)$  distribution, where  $\sigma^2$  is unknown. Derive the most powerful test of size  $\alpha = 0.05$  for testing  $H_0: \sigma^2 = 4$  against  $H_1: \sigma^2 = 1$ .

34. Let  $X_1, X_2, \dots, X_n$  be a random sample from a continuous distribution with the probability density function;

$$f(x; \lambda) = \begin{cases} \frac{2x}{\lambda} e^{-\frac{x^2}{\lambda}}, & \text{if } x > 0, \\ 0, & \text{otherwise,} \end{cases}$$

where  $\lambda > 0$ . Find the maximum likelihood estimator of  $\lambda$  and show that it is sufficient and an unbiased estimator of  $\lambda$ .